

Minimal Kochen-Specker theorem in finite dimensions

Sixia Yu^{1,2} and C.H. Oh¹

¹*Centre for Quantum Technologies and Physics Department,
National University of Singapore, 2 Science Drive 3, Singapore 117542*

²*Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics,
University of Science and Technology of China, Hefei, Anhui 230026, P.R. China*

In [1] we proved a strengthened Kochen-Specker theorem in 3 dimensions: non-contextual hidden variable (NCHV) models cannot reproduce all the quantum correlations of two compatible observables, which is a minimal requirement imposed on the NCHV models. Here we shall exclude the NCHV models with this minimal requirement in $d \geq 4$ dimensions by state-independent and experimentally testable inequalities satisfied by all NCHV models that are required to reproduce only the quantum correlations of at most two compatible observables. Furthermore our proofs use the smallest number of rays known so far, e.g., 25 (instead of 31) rays in 5 dimensions and $5d - 2\lfloor d/3 \rfloor$ rays in $d \geq 6$ dimensions.

Non-contextuality is a typical classical property: the value of an observable revealed by a measurement is predetermined regardless of which compatible observable might be measured alongside. A maximal set of mutually compatible observables defines a context. Kochen and Specker [2], and also Bell [3], showed that quantum mechanics (QM) is contextual by excluding all the non-contextual hidden variable (NCHV) models for a quantum system with more than 2 distinguishable states, known as Kochen-Specker (KS) theorem, via some kinds of logical contradictions. Nowadays the quantum contextuality becomes experimentally testable due to various KS inequalities [4–6], inequalities obeyed by all the NCHV models, and has been confirmed in recent experiments on different physical systems [7–14].

Originally KS theorem imposes a rather strong restriction on the NCHV models it excludes [2]: the partial algebraic structures of compatible observables in QM must be preserved. That is to say the NCHV models it excludes must admit the so-called *KS value assignment* in which the value assigned to the product or the sum of two compatible observables is equal to the product or the sum of the values assigned to these two compatible observables. Indeed the KS theorem is normally proved by finding a finite set of rays, called *KS proof*, to which the KS value assignment is impossible. The original KS proof in $d = 3$ dimensions includes 117 rays [2] which was reduced to 33 rays by Peres [15] and Schütte [16] and to 31 rays by Conway and Kochen [17]. In 4 dimensions Cabello, Estebaranz, and García-Alcaine (CEG) discovered a 18-ray proof [18]. KS proofs in higher dimensions can be constructed from those in lower dimensions by using either the Zimba-Penrose method [21], which is valid in $d > 5$ dimensions, or the CEG method [20]. There are also many state-dependent KS proofs, e.g., Clifton’s 8-ray proof [19] and the 5-ray proof in 3 dimensions [4], in which some rays have pre-assigned values.

It was Peres [22] who first noticed that the algebraic structures of compatible observables in QM need not to be completely preserved by NCHV models. This type of

KS proofs include Mermin-Peres’s magic-square proof in 4 dimensions [22] and Mermin’s pentagon [23] in 8 dimensions, in which the sum rule is abandoned. It is due to recently discovered KS inequalities that the restriction is lifted (almost) completely. Instead the NCHV models should reproduce the quantum correlations of at least three compatible observables (as in the KS inequalities of Peres-type), which demands that *three or more pairwise compatible observables be measured simultaneously*. QM has this property but it may not hold in a most general NCHV model since a context is well defined by pairwise compatibility. Thus a slight trail of algebraic structures of compatible observables in QM is still implicitly required to be preserved by the NCHV models excluded by KS inequalities.

Quite recently [1] we proved the KS theorem in 3 dimensions with a 13-ray set for which the KS value assignments do exist but fail to reproduce a certain prediction of QM. Based on this set a magic-cube inequality is discovered to exclude all those NCHV models that have only to reproduce the correlation of at most two compatible observables. This is a minimum number of compatible observables whose correlations have to be reproduced since for non-sequential measurements (or expectation of single observable) there does exist a NCHV model [2]. The question remains open whether there also exist such KS proofs in higher dimensions, given the numerical nature of the derivation of the magic-cube inequality. In this Letter we provide state-independent KS inequalities in all finite dimensions ($d \geq 4$) satisfied by NCHV models that have only to reproduce the correlations of at most two compatible observables. Moreover our proofs involve a smaller number of rays in all finite dimensions ($d \geq 5$) comparing with the best KS proofs known.

18-ray improved KS inequality in 4 dimensions— Motivated by Cabello’s 18-ray KS proof [18] we consider the 9-vertex graph shown in Fig.1. If we can associate each edge $e \in E$, the set of unordered pairs of connected vertices, with a 4-dimensional ray v_e in such a way that *two edges sharing a common vertex are associated with two*

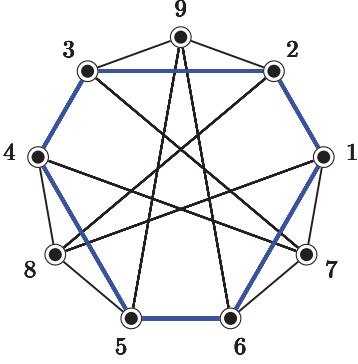


FIG. 1: (Color online) The orthogonality graph for 18 rays in Eq.(1), which are represented by edges instead of vertices with two rays sharing a common vertex being orthogonal.

orthogonal rays, i.e., two rays v_e and $v_{e'}$ with $e, e' \in E$ being different are orthogonal whenever $e \cap e' \neq \emptyset$, then we have a KS proof. Indeed the KS value assignment to this 18-ray set amounts to covering all the 9 vertices with disjoint edges. This is impossible because a set of disjoint edges must cover an even number of vertices while we have an odd number of vertices. It turns out that Cabello's 18-ray set in 4 dimensions as presented in [5]

$$\begin{aligned}
 v_{12} &= (1, 0, 0, 0) & v_{16} &= (0, 0, 1, \bar{1}) & v_{34} &= (\bar{1}, 1, 1, 1) \\
 v_{18} &= (0, 1, 0, 0) & v_{17} &= (0, 0, 1, 1) & v_{37} &= (1, 1, 1, \bar{1}) \\
 v_{28} &= (0, 0, 0, 1) & v_{67} &= (1, \bar{1}, 0, 0) & v_{47} &= (1, 1, \bar{1}, 1) \\
 v_{45} &= (0, 1, 0, \bar{1}) & v_{23} &= (0, 1, \bar{1}, 0) & v_{56} &= (1, 1, 1, 1) \\
 v_{48} &= (1, 0, 1, 0) & v_{29} &= (0, 1, 1, 0) & v_{59} &= (1, \bar{1}, 1, \bar{1}) \\
 v_{58} &= (1, 0, \bar{1}, 0) & v_{39} &= (1, 0, 0, 1) & v_{69} &= (1, 1, \bar{1}, \bar{1})
 \end{aligned} \quad (1)$$

is the unique set that satisfies the aforementioned orthogonality conditions up to a global unitary transformation to all the rays (see Appendix).

The first state-independent experimentally testable KS inequality [5] is built upon the above 18-ray set and the quantum correlations of four compatible observables have to be reproduced by the NCHV models it excludes. The KS inequality arising from Peres-Mermin's magic-square proof [5] still needs to check the quantum correlations of three compatible observables. In what follows we shall provide a KS inequality in 4 dimensions that involves the correlations of at most two compatible observables. For 18 binary variables $\{\bar{v}_e | e \in E\}$ taking values 0 and 1 the following algebraic inequality holds

$$\bar{L}_4 := \sum_{e \in E} \bar{v}_e - \frac{1}{2} \sum_{\substack{e \neq e' \in E \\ e \cap e' \neq \emptyset}} \bar{v}_e \cdot \bar{v}_{e'} \leq 4. \quad (2)$$

In fact if we denote $\bar{v}_i = \sum_{e \ni i} \bar{v}_e$ then we can rewrite $\bar{L}_4 = \sum_{i=1}^9 \bar{v}_i(2 - \bar{v}_i)/2$ taking account of the property $\bar{v}_e^2 = \bar{v}_e$. Because the quadratic function $x(2 - x)$ of x reaches its maximal value 1 at $x = 1$ and $\sum_i \bar{v}_i = 2 \sum_{e \in E} \bar{v}_e$ is an even number, not all 9 terms in \bar{L}_4 can

reach its maximum. That is to say for at least one $1 \leq i \leq 9$ it holds $\bar{v}_i(2 - \bar{v}_i) \leq 0$, from which the inequality Eq.(2) follows immediately. Since the last term in Eq.(2) is non-positive we have $\bar{L}_4 \leq \bar{E} := \sum_{e \in E} \bar{v}_e$. Also we shall denote by L_4 the theory-independent KS expression obtained from Eq.(2) with \bar{v} replaced by v .

In any NCHV model all observables have definite values determined only by some hidden variables λ that are distributed according to a given probability distribution ϱ_λ with normalization $\int d\lambda \varrho_\lambda = 1$. The average of an observable is given by $\langle v \rangle = \int d\lambda \varrho_\lambda v^\lambda$ while the correlation of two observables reads $\langle v \cdot u \rangle = \int d\lambda \varrho_\lambda v^\lambda u^\lambda$. As a result in any NCHV model it holds $\langle L_4 \rangle_c := \int d\lambda \varrho_\lambda L_4^\lambda \leq 4$, where L_4^λ is obtained from Eq.(2) by replacing \bar{v} with corresponding value v^λ determined by the hidden variables. Since each correlation is for compatible observables (orthogonal projections) the average $\langle L_4 \rangle_q := \text{Tr}(\rho \hat{L}_4)$ is well-defined in QM for an arbitrary state ρ of a 4-level system. Here the observable \hat{L}_4 is obtained from Eq.(2) by replacing \bar{v} with corresponding 1-dimensional projection $\hat{v} = |v\rangle\langle v| / \langle v|v \rangle$ in which $|v\rangle = v_1|0\rangle + v_2|1\rangle + v_3|2\rangle + \dots$ for a ray $v = (v_1, v_2, v_3, \dots)$. However in QM it holds the identity $\hat{L}_4 = q_4 I_4$ with $q_4 = 9/2$ and therefore in any state the quantum mechanics predicts $\langle L_4 \rangle_q = q_4 > 4$.

Furthermore if we require the NCHV models to admit a KS value assignment then we have $\bar{v}_i = \sum_{e \ni i} \bar{v}_e = 1$ for each vertex i and the last term in Eq.(2) vanishes. Choose any triple of unconnected vertices $\{i, j, k\}$, e.g., $\{7, 8, 9\}$, and denote $v_{ijk} := E - v_i - v_j - v_k$, e.g., $v_{789} = v_{12} + v_{23} + v_{34} + v_{45} + v_{56} + v_{16}$, which always form a hexagon (e.g. as shown in Fig.1 as thick blue lines). Inequality Eq.(2) becomes now $\bar{v}_{ijk} \leq 1$ so that $\langle v_{ijk} \rangle_c \leq 1$, which is referred to as *hexagon inequality*. While in QM we have identity $\hat{v}_{ijk} = 3/2$ so that in any state it holds $\langle v_{ijk} \rangle_q = 3/2 > \langle v_{ijk} \rangle_c$. Thus in order to exclude NCHV models with KS value assignments for 4-level system the hexagon inequality provides an experimental test that explicitly involves only 6 projections (out of 18). We note that the hexagon inequality and inequality Eq.(2) exclude different kinds of NCHV models.

25-ray KS inequality in 5 dimensions— Consider the orthogonality graph Δ_{13} shown in Fig.2 as a subgraph on 13 vertices labeled with lowercase letters. As shown in [1] the graph Δ_{13} determines uniquely, up to a global unitary transformation, the 13-ray set $V = y \cup h \cup z$ in 3 dimensions with $z = \{z_k = |k\rangle | k = 1, 2, 3\}$, $y = \{y_k^\sigma = |i\rangle + \sigma|j\rangle | (i, j, k) = 1, 2, 3; \sigma = \pm\}$, and $h = \{h_\alpha = \sum_{k=1}^3 (-1)^{\delta_{\alpha k}} |k\rangle | \alpha = 0, 1, 2, 3\}$. Let Γ be the adjacency matrix of Δ_{13} , a 13×13 symmetric matrix with vanishing diagonal elements. And $\Gamma_{uv} = 1$ if two vertices $u, v \in V$ are neighbors and $\Gamma_{uv} = 0$ otherwise. For 13 binary variables $\{\bar{v} | v \in V\}$ that take values 0 and 1 it holds

$$\bar{L}_3 := \bar{y} + \frac{\bar{h}}{2} + \bar{z} - \frac{1}{2} \sum_{u, v \in V} \Gamma_{uv} \bar{u} \cdot \bar{v} \leq \frac{7}{2}, \quad (3)$$

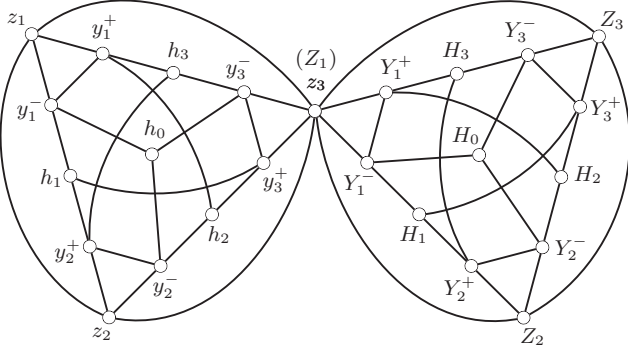


FIG. 2: The orthogonality graph for 25 rays in 5 dimensions with rays represented by vertices (hollow dots) and edges, straight or curved, linking two orthogonal rays. For the sake of clarity not all orthogonality relations are shown.

where $\bar{h} = \sum_{v \in h} \bar{v}$ for any subset $h \subseteq V$. In fact it is exactly the same inequality as in [1] if we make a change of variables $a_v = 1 - 2v$. Therefore in any NCHV model it holds $\langle L_3 \rangle_c \leq c_3 := 7/2$ while for all qutrit states it holds $\langle \hat{L}_3 \rangle_q = q_3 > c_3$ with $q_3 = 11/3$ because of the identity $\hat{L}_3 = q_3 I_3$. For later use we note that it always holds $\bar{L}_3 \leq \bar{V} = \bar{y} + \bar{h} + \bar{z}$.

According to CEG method [20] we consider the 25-ray set $V_5 = V_+ \cup V_-$ in 5 dimensions with $V_+ = \{(v, 0, 0) | v \in V\}$ and $V_- = \{(0, 0, v) | v \in V\}$. For simplicity we shall use the same lowercase letters to represent 5-dimensional rays in V_+ , e.g., z_1 for $(z_1, 0, 0)$, while the uppercase letters to represent the corresponding rays in V_- , e.g., Z_1 for $(0, 0, z_1)$. Since $V_+ \cap V_- = \{z_3 = Z_1\}$ we have indeed $|V_5| = 25$ rays. Part of the orthogonality relations among these 25 rays are shown in Fig.2. In addition rays labeled with $z_{1,2}$ ($Z_{2,3}$) are orthogonal to all the rays labeled with uppercase (lowercase) letters and rays.

Let L_3^\pm be the corresponding KS expression as defined in Eq.(3) for two 13-ray sets V_\pm and denote $\bar{z}' = \bar{z}_{12} + \bar{Z}_{23}$ where $\bar{z}_{12} = \bar{z}_1 + \bar{z}_2$ and $\bar{Z}_{23} = \bar{Z}_2 + \bar{Z}_3$. For 2^{25} possible values of 25 binary variables $\{\bar{v} | v \in V_5\}$ taking values 0 or 1 it holds the following inequality

$$\bar{L}_5 := \bar{L}_3^+ + \bar{L}_3^- + \frac{11}{3} \bar{z}' (2 - \bar{z}') - \bar{z}_{12} \cdot \bar{V}_- - \bar{Z}_{23} \cdot \bar{V}_+ \leq \frac{43}{6}, \quad (4)$$

where we have denoted $\bar{V}_\pm = \sum_{v \in V_\pm} \bar{v}$. To prove the

inequality above we have only to consider the following four cases by noting that \bar{z}_{12} and \bar{Z}_{23} are non-negative integers: i) if $\bar{z}_{12} = \bar{Z}_{23} = 0$ then \bar{z}' also vanishes so that $\bar{L}_5 \leq 2c_3 = 7$; ii) if $\bar{z}_{12} = 0$ while $\bar{Z}_{23} \geq 1$ then we have $\bar{L}_5 \leq q_3 + \bar{L}_3^- \leq c_3 + q_3 = 43/6$ since $\bar{L}_3^+ \leq \bar{V}_+$ and $x(2-x) \leq 1$; iii) the same bound holds also in the case of $\bar{z}_{12} \geq 1$ while $\bar{Z}_{23} = 0$ for the same reason as case ii); iv) if $\bar{z}_{12} \geq 1$ and $\bar{Z}_{23} \geq 1$ we have $\bar{L}_5 \leq 11/3$ since $\bar{L}_3^\pm \leq \bar{V}_\pm$ and $x(2-x) \leq 1$. If we replace \bar{v} by corresponding 1-dimensional projection the quantum mechanical prediction reads $\hat{L}_5 = 2q_3 I_5$ and thus in any state it holds $\langle L_5 \rangle_q = 22/3 > 43/6$.

In NCHV models admitting KS value assignments, it holds $\bar{y} + \bar{z} + 3\bar{Z}_{23} = 3$ and $\bar{Y} + \bar{Z} + 3\bar{z}_{12} = 3$ and all quadratic terms in Eq.(4) vanish. As a result the KS inequality Eq.(4) now becomes $\bar{L}_5' := (\bar{h} + \bar{H})/2 + 2\bar{z}'/3 \leq 7/6$ and thus $\langle L_5' \rangle_c \leq 7/6$ must be satisfied by all NCHV models with KS value assignments. While quantum mechanically it holds identity $\hat{L}_5' = \frac{4}{3} I_5$, meaning that we have a violation $\langle L_5' \rangle_q = 4/3 > 7/6$ state-independently. Here only 12 projections, namely $\{h_\alpha\}$ and $\{H_\alpha\}$ together with $z_{1,2}$ and $Z_{2,3}$, are involved explicitly.

Improved KS inequality in $d \geq 6$ dimensions— Consider now $d \geq 6$ and there always exist two nonnegative integers m, n such that $d = 3m + 4n$. Accordingly the qudit Hilbert space H_d can be decomposed into a direct sum of m qutrit Hilbert spaces H_3^k and n 4-dimensional Hilbert spaces H_4^l as $H_d = \oplus_{k=1}^m H_3^k \oplus_{l=1}^n H_4^l$. According to Zimba-Penrose method [21] we obtain a set $V_d = \cup_k V_k \cup_l E_l$ of $13m + 18n$ rays in d dimensions. Here V_k is the 13-ray set in d dimensions that is supported only on the qutrit Hilbert space H_3^k , i.e., obtained from the 13-ray set V by appending necessary zeros, and E_l is the 18-ray set in Eq.(1) that is supported only on the subspace H_4^l . By construction, each ray in V_k is orthogonal to all the rays in $V_{k'}$ for $k \neq k'$. Also each ray in V_k is orthogonal to all the rays in E_l and every ray in E_l is orthogonal to all the rays in $E_{l'}$ for $l \neq l'$.

Consider binary variables $\{\bar{v} | v \in V_d\}$ taking values 0 or 1 and let $L_3^{(k)}$ and $L_4^{(l)}$ be the KS expressions Eq.(3) for the 13-ray set V_k and Eq.(2) for the 18-ray set E_l , respectively, and $\bar{E}_l = \sum_{v \in E_l} \bar{v}$ and $\bar{V}_k = \sum_{v \in V_k} \bar{v}$. Then we have the following algebraic inequality

$$\bar{L}_d := \frac{1}{q_3} \sum_{k=1}^m \bar{L}_3^{(k)} + \frac{1}{q_4} \sum_{l=1}^n \bar{L}_4^{(l)} - \sum_{k=1}^m \bar{V}_k \cdot \sum_{l=1}^n \bar{E}_l - \frac{1}{q_3} \sum_{\substack{k > k' \\ k, k'=1}}^m \bar{V}_k \cdot \bar{V}_{k'} - \frac{1}{q_4} \sum_{\substack{l > l' \\ l, l'=1}}^n \bar{E}_l \cdot \bar{E}_{l'} \leq \frac{c_3}{q_3}. \quad (5)$$

In fact by noting that $\bar{V}_T = \sum_k \bar{V}_k$ and $\bar{E}_T = \sum_l \bar{E}_l$

are integers we have only to consider the following four

d	r_2	r_d	
3	13	31	
5	25	29	
$4m$	$18m - 2 \lfloor m/3 \rfloor$	$18m$	$(m \geq 1)$
$4m + 5$	$18m + 23 - 2 \lfloor (m+2)/3 \rfloor$	$18m + 25$	$(m \geq 1)$
$4m + 6$	$18m + 26 - 2 \lfloor m/3 \rfloor$	$18m + 31$	$(m \geq 0)$
$4m + 7$	$18m + 31 - 2 \lfloor (m+1)/3 \rfloor$	$18m + 34$	$(m \geq 0)$

TABLE I: A summary of the number r_2 of rays used in the improved KS proof in d dimensions in comparison with the smallest known number r_d of rays taken from [20] in dimensions 5,6, and 7 which are extended to dimensions $4m + \{5, 6, 7\}$ using Zimba-Penrose method.

cases: i) $\bar{V}_T, \bar{E}_T \geq 1$ ii) $\bar{V}_T \geq 1$ while $\bar{E}_T = 0$; iii) $\bar{E}_T \geq 1$ while $\bar{V}_T = 0$; iv) $\bar{E}_T = \bar{V}_T = 0$. In the first case where $\bar{V}_T, \bar{E}_T \geq 1$ we obtain $q_3 q_4 \bar{L}_d \leq 1 - (1 - q_3 \bar{E}_T)(1 - q_4 \bar{V}_T) \leq 1$, i.e., $\bar{L}_d \leq 1/(q_3 q_4) < c_3/q_3$ by neglecting the last term and taking into account $\bar{L}_k \leq \bar{V}_k$. In the second case it follows from $\bar{E}_T = 0$, i.e., $\bar{E}_l = 0$ for all $l = 1, 2, \dots, n$ and $\bar{V}_T \geq 1$ that $\bar{L}_4^{(l)} \leq \bar{E}_l = 0$ for all l . If there is one and only one nonzero \bar{V}_k , say $\bar{V}_1 \geq 1$ and $\bar{V}_k = 0$ for $k \neq 1$, then $\bar{L}_d \leq \bar{L}_3^{(1)}/q_3 \leq c_3/q_3$. If there are exactly two nonzero \bar{V}_k 's, say $\bar{V}_1 \geq 1$ and $\bar{V}_2 \geq 1$, we have $q_3 \bar{L}_d \leq \bar{V}_1 + \bar{V}_2 - \bar{V}_1 \bar{V}_2 \leq 1$ i.e., $\bar{L}_d \leq 1/q_3 < c_3/q_3$. If there are more than three nonzero \bar{V}_k 's, say $\bar{V}_k \geq 1$ for $k = 1, 2, \dots, K \geq 3$ then $\sum_{k \geq k'} \bar{V}_k \bar{V}_{k'} \geq \sum_k \bar{V}_k \bar{V}_{k+1} \geq \sum_k \bar{V}_k$ in which we have identified $K+1$ with 1. As a result $\bar{L}_d \leq 0$. In the third case it follows from $\bar{V}_T = 0$, i.e., $\bar{V}_k = 0$ for all $k = 1, 2, \dots, m$ and $\bar{E}_T \geq 1$, that $\bar{L}_3^{(k)} \leq \bar{V}_k = 0$ for all k . Furthermore if there is one and only one nonzero \bar{E}_l , say $\bar{E}_1 \geq 1$, then $\bar{L}_4^{(l)} \leq 0$ except $l = 1$ so that $\bar{L}_d \leq \bar{L}_4^{(1)}/q_4 \leq 4/q_4 < c_3/q_3$. If there are two and only two nonzero \bar{E}_l 's, say $\bar{E}_1 \geq 1$ and $\bar{E}_2 \geq 1$, then $q_4 \bar{L}_d \leq \bar{E}_1 + \bar{E}_2 - \bar{E}_1 \bar{E}_2 \leq 1$, i.e., $\bar{L}_d \leq 1/q_4 < c_3/q_3$. If there are three or more $\bar{E}_l \geq 1$ for $l \in K$ then it holds $\sum_{l \in K} \bar{E}_l \leq \sum_{l \neq l' \in K} \bar{E}_l \bar{E}_{l'}$ so that $\bar{L}_d \leq 0$. Finally if $\bar{E}_T = \bar{V}_T = 0$ then $\bar{L}_d \leq 0$. Thus in all cases we have $\bar{L}_d \leq c_3/q_3$. In any NCHV model it therefore holds $\langle L_d \rangle_c \leq c_3/q_3$. However in QM we have identity $\hat{L}_d = I_d$ from which it follows that in any qudit state $\langle L_d \rangle_q = 1 > c_3/q_3$.

If two integers m_0, n_0 is a decomposition of $d = 3m_0 + 4n_0$ then $m = m_0 + 4l$ and $n = n_0 - 3l$ is also a decomposition of $d = 3m + 4n$ for arbitrary l and in this case a number of $13m_0 + 18n_0 - 2l$ rays is required. If we choose l optimally, i.e., $m = 4\lfloor d/3 \rfloor - d$ and $n = d - 3\lfloor d/3 \rfloor$ where $\lfloor x \rfloor$ denotes the largest integer that is no greater than x , then a number $r_2 = 5d - 2\lfloor d/3 \rfloor$ of rays is involved. Our results are summarized in Table 1 in which the smallest numbers r_d of rays used in known KS proofs in various dimensions are presented for comparison. For

the NCHV models with KS value assignment the inequality Eq.(5) can be simplified to an inequality that involves $2(d - \lfloor d/3 \rfloor)$ projections explicitly.

Our improved KS inequalities derived above, despite of the fact that the properties of discrete nature of binary observables (taking value 0,1) have been used, still hold true for variables taking values between 0 and 1 continuously. In fact all the inequalities considered above involve non-definite quadratic forms of binary variables, which arise from the adjacency matrices of graphs. Therefore the extremal values must be attained on the boundaries which is exactly what have been considered above.

To summarize, we have derived state-independent KS inequalities in $d \geq 4$ dimensions that involve the correlations of at most two compatible observables. And thus NCHV models with only a minimal requirement, in which three or more pairwise compatible observables may even not be simultaneously measurable, cannot reproduce the quantum mechanical predictions on the correlations of two compatible observables, which can be called suitably as a minimal KS theorem. Moreover our improved KS inequalities involve a smaller number of rays than all the KS proofs previously known and therefore are more accessible to experimental tests.

Acknowledgement— This work is supported by National Research Foundation and Ministry of Education, Singapore (Grant No. WBS: R-710-000-008-271) and NSF of China (Grant No. 11075227).

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Appendix— We shall prove in what follows that up to a global unitary transformation the orthogonality graph in Fig.1 uniquely determines the 18-ray set. The orthogonality conditions read: *two rays labeled with two edges with a common vertex should be orthogonal*. Without loss of generality we can assume

$$\left. \begin{aligned} v_{12} &= (1, 0, 0, 0) \\ v_{18} &= (0, 1, 0, 0) \end{aligned} \right\} v_{28} = (0, 0, b, \bar{1})$$

$$\left. \begin{aligned} v_{17} &= (0, 0, 1, 0) \\ v_{16} &= (0, 0, 0, 1) \end{aligned} \right\} v_{67} = (1, \bar{a}, 0, 0)$$
(A.1)

with $a, b > 0$ because i) any quadruplet of rays sharing a single vertex, e.g., vertex 1, should form a basis; ii) v_{28} should be orthogonal to both v_{12} and v_{18} ; iii) complex numbers a and b can be chosen to be positive by multiplying a suitable phase factor to v_{18} and v_{17} respectively. Taking into account of quadruplets of rays having common vertices 2, 8, 7, and 6 we have

$$\left. \begin{aligned} v_{29} &= (0, u_2, 1, b) \\ v_{23} &= (0, u'_2, 1, b) \end{aligned} \right\} v_{39} = (g, 0, b, \bar{1})$$

$$\left. \begin{aligned} v_{48} &= (u_8, 0, 1, b) \\ v_{58} &= (u'_8, 0, 1, b) \end{aligned} \right\} v_{45} = (0, h, b, \bar{1})$$

$$\left. \begin{aligned} v_{37} &= (a, 1, 0, u_7) \\ v_{47} &= (a, 1, 0, u'_7) \end{aligned} \right\} v_{34} = (1, \bar{a}, s, 0)$$

$$\left. \begin{aligned} v_{56} &= (a, 1, u_6, 0) \\ v_{69} &= (a, 1, u'_6, 0) \end{aligned} \right\} v_{59} = (1, \bar{a}, 0, t)$$
(A.2)

for some nonzero complex numbers g, h, s, t and u_k, u'_k satisfying $-u'_k u_k^* = 1 + a^2$ for $k = 6, 7$ and $-u'_k u_k^* = 1 + b^2$ for $k = 2, 8$. Furthermore we can choose t to be real and $t > 0$ by multiplying the same suitable phase factor to v_{12} and v_{18} . From the orthogonality of two quadruplets of rays sharing the vertex 3 and 9

$$\begin{aligned} v_{23} &= (0, u'_2, 1, b) & v_{45} &= (0, h, b, \bar{1}) \\ v_{34} &= (1, \bar{a}, s, 0) & v_{56} &= (a, 1, u_6, 0) \\ v_{37} &= (a, 1, 0, u_7) & v_{58} &= (u'_8, 0, 1, b) \\ v_{39} &= (g, 0, b, \bar{1}) & v_{59} &= (1, \bar{a}, 0, t) \end{aligned}$$
(A.3)

it follows immediately

$$a = \frac{s^*}{u'_2} = \frac{u_7}{g^*} = -\frac{t}{h} = -\frac{u_6^*}{u'_8}, \quad (A.4)$$

$$b = -\frac{u'_2}{u_7^*} = -\frac{g^*}{s} = -\frac{h^*}{u_6} = -\frac{u'_8}{t}. \quad (A.5)$$

As one result we obtain, keeping in mind that a, b are real, $ab = -s/u_7 = -u_7/s$ and thus $ab = 1$. From the orthogonality of two quadruplets of rays sharing the vertex 4 and 5

$$\begin{aligned} v_{29} &= (0, u_2, 1, b) & v_{34} &= (1, \bar{a}, s, 0) \\ v_{39} &= (g, 0, b, \bar{1}) & v_{45} &= (0, h, b, \bar{1}) \\ v_{59} &= (1, \bar{a}, 0, t) & v_{47} &= (a, 1, 0, u'_7) \\ v_{69} &= (a, 1, u'_6, 0) & v_{48} &= (u_8, 0, 1, b) \end{aligned}$$
(A.6)

it follows $u'_6 = -u_2^*$, $g = t$, and $u_8 = -s^*$, $u'_7 = h^*$ together with

$$\frac{a}{b} = \frac{t}{u_2} = -\frac{u'_6}{g^*} = -\frac{u'_7}{u_8^*} = \frac{s^*}{h}. \quad (A.7)$$

Consequently we obtain $a/b = t/u_2 = u_2/t$ so that $a/b = 1$ and thus $a = b = 1$. As a result $h = s = -t$ and u_k are all real with $u_k = -u'_k$ and $u_2 = u_7 = u_6 = u_8 = t$ and therefore $t^2 = 2$. A rotation, $(1, 0) \mapsto (1, 1)/\sqrt{2}$ and $(0, 1) \mapsto (1, \bar{1})/\sqrt{2}$ in the subspace spanned by v_{16} and v_{17} recovers Cabello's 18-ray set Eq.(1) exactly.